

EISENSTEIN–TYPE SERIES ASSOCIATED TO PARTITION RANKS

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ABSTRACT. In this paper, we introduce a class of functions that behave like classical Eisenstein series in many ways, but with a key distinction: only their non-holomorphic completions transform like (quasi)modular forms. We show how the partition rank generating function can be expressed in terms of partition traces of these functions. A key feature of our construction is that the completions satisfy a holomorphic anomaly equation—a phenomenon typically seen in the context of quantum field theory and string theory. We also show that the Fourier coefficients of these Eisenstein-type series are integral.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper, we introduce a new class of Eisenstein-type series. These series share several structural features with classical Eisenstein series, but differ in a crucial respect: only their non-holomorphic completions transform as (quasi)modular forms. To motivate their definition, we first explore their connection to the rank generating function, before examining their further properties in detail.

1.1. Background. A *partition* of $n \in \mathbb{N}_0$ is a weakly decreasing sequence of positive integers that sum to n . We denote by $p(n)$ the number of partitions of n . Recall the famous Ramanujan congruences

$$p(5n + 4) \equiv 0 \pmod{5}, \quad p(7n + 5) \equiv 0 \pmod{7}, \quad p(11n + 6) \equiv 0 \pmod{11}.$$

To explain the first two, Dyson [11] introduced the *rank* of a partition λ , which is defined as

$$\text{rank}(\lambda) := \text{largest part of } \lambda - \text{number of parts of } \lambda.$$

Dyson conjectured that reducing the rank (mod 5) (resp. 7) divides the partitions of $5n + 4$ (resp. $7n + 5$) into 5 (resp. 7) sets of equal size. This conjecture was proven by Atkin and Swinnerton-Dyer [5]. In the same paper, Dyson also conjectured the existence of another statistic, which he called the “crank” and which should explain all three partition congruences. Garvan [15] found a crank for vector partitions and Andrews–Garvan [4] defined a crank for ordinary partitions. Letting $o(\lambda)$ denote the number of ones in a partition λ , and $\mu(\lambda)$ the number of parts strictly larger than $o(\lambda)$, the *crank* is defined as

$$\text{crank}(\lambda) := \begin{cases} \text{largest part of } \lambda & \text{if } o(\lambda) = 0, \\ \mu(\lambda) - o(\lambda) & \text{if } o(\lambda) > 0. \end{cases}$$

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Let $N(m, n)$ denote the number of partitions of n with rank m . Its generating function is

$$R(\zeta; q) := \sum_{\substack{n \geq 0 \\ m \in \mathbb{Z}}} N(m, n) \zeta^m q^n = \sum_{n \geq 0} \frac{q^{n^2}}{(\zeta q)_n (\zeta^{-1} q)_n}, \quad (1.1)$$

(see [5]) where $(a)_n = (a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j)$ for $n \in \mathbb{N}_0 \cup \{\infty\}$. Let $M(m, n)$ denote the number of partitions of n with crank m , except for¹ $n = 1$ where $M(-1, 1) = -M(0, 1) = M(1, 1) := 1$ as given by the following generating function [4]

$$C(\zeta; q) := \sum_{\substack{n \geq 0 \\ m \in \mathbb{Z}}} M(m, n) \zeta^m q^n = \frac{(q)_\infty}{(\zeta q)_\infty (\zeta^{-1} q)_\infty}. \quad (1.2)$$

Modularity properties of the rank and of the crank generating function differ significantly: the crank generating function is basically a meromorphic Jacobi form, whereas the rank generating function is a “mock Jacobi form” [8]: it only transforms like a Jacobi form after adding a non-holomorphic term (see Subsection 2.3 for the precise transformations of the rank generating function). We next consider the *crank moments* [3]

$$C_k(q) := \sum_{\substack{n \geq 0 \\ m \in \mathbb{Z}}} m^k M(m, n) q^n.$$

These moments can be written in terms of quasimodular forms. In [1], the authors expressed these as a so-called partition Eisenstein trace. Here, for a sequence of functions $h = \{h_k\}_{k \in \mathbb{N}}$, define, for $n \in \mathbb{N}$, the n -th *partition trace* with respect to h and a function ϕ on partitions as

$$\mathrm{Tr}_n(\phi, h; \tau) := \sum_{\lambda \vdash n} \phi(\lambda) h_\lambda(\tau),$$

where the sum ranges over all partitions of n and for $\lambda = (1^{m_1}, 2^{m_2}, \dots, n^{m_n}) \vdash n$, we set

$$h_\lambda(\tau) := \prod_{j=1}^n h_j^{m_j}(\tau).$$

Partition Eisenstein traces are the partition traces with respect to the sequence of Eisenstein series $G = \{G_k\}_{k \in \mathbb{N}}$. Here, G_k is the *Eisenstein series*² of weight $k \in 2\mathbb{N}$, given by

$$G_k(\tau) := -\frac{B_k}{2k} + \sum_{n, m \geq 1} m^{k-1} q^{nm} \quad (q := e^{2\pi i \tau}, \tau \in \mathbb{H} := \{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}),$$

with B_k the k -th Bernoulli number. By convention, $G_k := 0$ for k odd. Moreover, define

$$\phi(\lambda) := \prod_{j=1}^k \frac{2^{m_j}}{m_j! j!^{m_j}}. \quad (1.3)$$

The following result was obtained in [1, Theorem 1.2].

¹The correct combinatorial values for the anomalous case $n = 1$ are 1 if $m = 0$ and 0 for $m \neq 0$.

²In [1] the authors wrote G_k for what here is $2G_k$ and $\mathrm{Tr}_k(\phi; \tau)$ for what is $\mathrm{Tr}_{2k}(\phi, G; \tau)$ here.

Theorem 1.1. *We have*

$$\sum_{k \geq 0} C_k(q) \frac{z^k}{k!} = \frac{2 \sinh(\frac{z}{2})}{z(q)_\infty} \sum_{k \geq 0} \text{Tr}_k(\phi, G; \tau) z^k.$$

The Eisenstein series G_k play a key role in the theory of modular forms and satisfy many interesting properties. They have a beautiful connection with Bernoulli numbers in that, for $k \geq 2$, we have

$$\lim_{\tau \rightarrow i\infty} G_k(\tau) = -\frac{B_k}{2k}.$$

For $k > 2$, G_k is a modular form of weight k on $\text{SL}_2(\mathbb{Z})$ and G_2 is quasimodular (see Subsection 2.1 for the definition). More precisely, we have for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$,

$$G_k \left(\frac{a\tau + b}{c\tau + d} \right) = \begin{cases} (c\tau + d)^k G_k(\tau) & \text{if } k \neq 2, \\ (c\tau + d)^2 G_2(\tau) + \frac{ic}{4\pi}(c\tau + d) & \text{if } k = 2. \end{cases} \quad (1.4)$$

Another key property of the Eisenstein series is that the corresponding algebra $\mathbb{Q}[G_2, G_4, G_6, \dots]$ of quasimodular forms is closed under the action of $D := q \frac{\partial}{\partial q}$.

1.2. Main results. A result like Theorem 1.1 involving the *rank moments*

$$R_k(q) := \sum_{\substack{n \geq 0 \\ m \in \mathbb{Z}}} m^k N(m, n) q^n \quad (1.5)$$

may give rise to an interesting generalization of the Eisenstein series. In this paper, we show that this is indeed the case. We obtain a surprisingly simple definition of Eisenstein-type series³ g_ℓ in (1.7) which seem not to be studied in the literature before. As mentioned above, the rank moments are related to mock modular forms. Hence, they do not admit an expansion as a trace of modular Eisenstein series like crank moments. As our first result, we write the rank moments as traces of the Eisenstein-type series f_k defined in Subsection 3.1. The aim of this paper is to understand these functions f_k . We call a real-analytic function $f^*(\tau, \bar{\tau})$ a *quasi-completion* of $f(\tau)$, if $f^*(\tau, \bar{\tau})$ transforms like a quasimodular form (see (2.2)) and if⁴ $\lim_{\bar{\tau} \rightarrow -i\infty} f^*(\tau, \bar{\tau}) = f(\tau)$. If a quasi-completion $f^*(\tau, \bar{\tau})$ transforms as a modular form, then we call $f^*(\tau, \bar{\tau})$ a *completion*. If it is clear from the context, then we also just write $f^*(\tau)$ instead of $f^*(\tau, \bar{\tau})$. Our first result is the following theorem.

Theorem 1.2. *There exists a family of functions $f = \{f_k\}_{k \in \mathbb{N}}$ such that*

$$\sum_{k \geq 0} R_k(q) \frac{z^k}{k!} = \frac{2 \sinh(\frac{z}{2})}{z(q)_\infty} \sum_{k \geq 0} \text{Tr}_k(\phi, f; \tau) z^k, \quad (1.6)$$

where ϕ is defined in (1.3) and f_k has the following properties:

(1) For $k \geq 2$, we have

$$\lim_{\tau \rightarrow i\infty} f_k(\tau) = -\frac{B_k}{2k}.$$

³We emphasize that these functions are not constructed by averaging, as is done for classical Eisenstein series. They are called Eisenstein-type because they satisfy properties analogous to those of classical Eisenstein series.

⁴Here and throughout, we consider τ and $\bar{\tau}$ as independent variables.

(2) The function f_k has a quasi-completion⁵ f_k^* which satisfies, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$f_k^* \left(\frac{a\tau + b}{c\tau + d} \right) = \begin{cases} (c\tau + d)^k f_k^*(\tau) & \text{if } k \neq 2, \\ (c\tau + d)^2 f_2^*(\tau) + \frac{3ic}{4\pi}(c\tau + d) & \text{if } k = 2. \end{cases}$$

(3) The algebra $\mathcal{F} := \mathbb{Q}[f_2, f_4, \dots, G_2, G_4, \dots]$ is closed under the action of D .

Remark.

- (1) As for the Eisenstein series, we have $f_k = 0$ if k is odd; see Remark 3.1.
- (2) The functions f_k and f_k^* are of intrinsic interest, as they are a step up in complexity compared to quasimodular forms.
- (3) The transformation of f_k^* agrees with that of G_k up to a factor of 3 in front of $c\tau + d$. This is explained by the (mock) Jacobi forms underlying the rank and crank statistics: namely, for the rank the corresponding index is $-\frac{3}{2}$ and for the crank it is $-\frac{1}{2}$.

Even though the f_k are uniquely determined in Theorem 1.2, properties (1), (2), and (3) itself do not determine them uniquely. For example, after adding a cusp form of weight k to each f_k , the resulting functions still satisfy these properties. To describe the f_k uniquely without involving (1.6), we give two recursive definitions for them involving divisor-like sums. For this, we define $g_\ell \in \mathcal{F}$ by $g_0 := 1$, $g_\ell := 0$ for $\ell \in \mathbb{N}$ odd, and, for $\ell \in \mathbb{N}$ even,

$$g_\ell(\tau) := \left(1 - 2^{\ell-1}\right) \frac{B_\ell}{2^\ell} + \sum_{2n-1 \geq 3m \geq 3} (2n-3m)^{\ell-1} q^{nm} - \sum_{n-1 \geq 6m \geq 6} (n-6m)^{\ell-1} q^{nm}. \quad (1.7)$$

For $r \in \mathbb{N}$, the r -th Fourier coefficient of g_ℓ is a polynomial in some of the divisors of r : namely those positive divisors n, m satisfying the inequality $2n - 1 \geq 3m$ or $n - 1 \geq 6m$. Hence, the g_k are in spirit of the ‘‘mock Eisenstein series’’ studied by Zagier [25, p.15] and Mertens–Ono–Rolen [19, equation (1.4)]. Moreover, compare g_ℓ with

$$F_\ell^{[2]}(\tau) := \sum_{\substack{n > m \geq 1 \\ n-m \text{ odd}}} (-1)^n m^{\ell-1} q^{\frac{nm}{2}}$$

which can be expressed as the Rankin–Cohen bracket of a mock modular form and η^3 as explained in [10, Example 4 on p. 44]. This function goes back to [12, 13], where the authors studied characters of the $\mathcal{N} = 4$ superconformal algebra in two dimensions and the elliptic genus of K3 surfaces.

Theorem 1.3. *Let $n \in \mathbb{N}$.*

(1) *We have*

$$f_n(\tau) = \frac{n}{2^{n-1}} g_n(\tau) - \sum_{\ell=2}^{n-2} \frac{\ell}{2^{\ell-2}} \binom{n-1}{\ell} f_{n-\ell}(\tau) g_\ell(\tau).$$

(2) *We have*

$$f_n(\tau) = \sum_{\ell=2}^n \frac{(n-1)! \ell}{(\ell-1)! 2^{\ell-1}} g_\ell(\tau) \mathrm{Tr}_{n-\ell}(\psi, f; \tau),$$

with $f := \{f_k(\tau)\}_{k \in \mathbb{N}}$ and $\psi(\lambda) := (-1)^{\sum_{j=1}^k m_j} \phi(\lambda)$.

⁵We consider quasi-completions, because they are more analogous to the functions in Theorem 1.1. In the sequel, we also study the completions \widehat{f}_k of f_k ; see (3.4) and (3.6) for their definitions.

As a direct corollary, we see that the algebra $\mathcal{G} := \mathbb{Q}[g_2, g_4, \dots, G_2, G_4, \dots]$ is closed under D , and that the elements of \mathcal{G} admit a quasi-completion:

Corollary 1.4. *We have $\mathcal{G} = \mathcal{F}$, where the algebra \mathcal{F} is given by Theorem 1.2 (3).*

The recursive formulas in Theorem 1.3 suggest that the Fourier coefficients of f_k may have large denominators. However, we show in the following theorem that, with the exception of the constant term, all Fourier coefficients of f_k are integers, similar as for G_k .

Theorem 1.5. *For $k \geq 2$, the Fourier coefficients of $f_k + \frac{B_k}{2k}$ are integers.*

Finally, in part (1) of the next theorem, we give an explicit formula for $D(f_k)$ and explain how the raising and lowering operators act on the completions \widehat{f}_k of the⁶ f_k . More precisely, let $\widehat{\mathcal{F}} := \mathbb{C}[\widehat{f}_2, \widehat{f}_4, \widehat{f}_6, \dots, \widehat{G}_2, G_4, G_6]$ and define the *raising* and the *lowering operator* by

$$\mathcal{R}_k := 2i \frac{\partial}{\partial \tau} + \frac{k}{v}, \quad L := -2iv^2 \frac{\partial}{\partial \bar{\tau}} \quad (k \in \mathbb{Z}, \tau = u + iv).$$

In parts (2) and (3), we obtain a recursive expression for the action of the raising and lowering operators on \widehat{f}_k . In physics literature, the latter of these falls into the realm of “holomorphic anomaly equations”. In string theory, mirror symmetry, enumerative geometry, and topological field theory, a *holomorphic anomaly* refers to the phenomenon where a generating function (or its coefficients), instead of being purely holomorphic, acquires a controlled dependence on both τ and $\bar{\tau}$. A holomorphic anomaly equation describes this controlled dependency through a recursive relation that expresses the anti-holomorphic variation in terms of functions of lower genus or weight. While these may themselves carry non-holomorphic dependence, the anomaly equation organizes this structure in a precise and constrained way (see for example [16, 17, 18, 20]). We let $\delta_{\mathcal{S}} := 1$ if a statement \mathcal{S} holds and $\delta_{\mathcal{S}} := 0$ otherwise.

Theorem 1.6. *We have the following.*

(1) *For $k \geq 2$, we have*

$$D(f_k(\tau)) = \frac{k!}{6} \text{Tr}_{k+2}(\phi, 3G - f; \tau) - \frac{k-1}{6(k+1)} f_{k+2}(\tau) - \frac{1}{3} \sum_{a=1}^{k-1} \binom{k}{a} f_{a+1}(\tau) f_{k-a+1}(\tau).$$

(2) *The algebra $\widehat{\mathcal{F}}$ is closed under the raising operator. In particular, for $k \geq 2$ we have*

$$\begin{aligned} -\frac{1}{4\pi} \mathcal{R}_k(\widehat{f}_k(\tau)) &= \frac{k!}{6} \text{Tr}_{k+2}(\phi, 3\widehat{G} - \widehat{f}; \tau) - \frac{k-1}{6(k+1)} \widehat{f}_{k+2}(\tau) \\ &\quad - \frac{1}{3} \sum_{a=1}^{k-1} \binom{k}{a} \widehat{f}_{a+1}(\tau) \widehat{f}_{k-a+1}(\tau), \end{aligned}$$

where $\widehat{G} = \{\widehat{G}_k\}_{k \geq 1}$ is defined by $\widehat{G}_k = G_k$ for $k \neq 2$ and \widehat{G}_2 is the completion of G_2 , given in (2.1).

(3) *We have $L(\widehat{\mathcal{F}}) \subseteq \widehat{\mathcal{F}} \oplus \sqrt{v}|\eta|^2 \widehat{\mathcal{F}}$. In particular, for $k \geq 2$ we have*

$$L(\widehat{f}_k(\tau)) = -\frac{3}{8\pi} \delta_{k=2} + \frac{\sqrt{3}k!}{4\sqrt{2}\pi} \sqrt{v}|\eta(\tau)|^2 \text{Tr}_{k-2}(\psi, \widehat{f}; \tau),$$

where ψ is defined in Theorem 1.3 (2).

⁶See Subsection 2.1 for the definition; in particular, \widehat{f}_k and f_k^* agree if $k \neq 2$.

(4) If $\widehat{f} \in \widehat{\mathcal{F}}$ is of weight k , then $(D + \frac{2k}{3}f_2^*)(\widehat{f})$ transforms modular of weight $k + 2$.

Remark. We expect that the algebra \mathcal{F} is freely generated by the f_k for k even and G_2, G_4 , and G_6 (see also question (2) in Section 6). In particular, this would imply that the subalgebra $\mathbb{Q}[f_2, f_4, \dots]$ of \mathcal{F} is not closed under applying D , since Theorem 1.6 (1) involves the modular Eisenstein series.

1.3. Structure of the paper. The paper is organized as follows. In Section 2, we provide certain preliminaries on quasimodular forms and crank moments, the completion of the rank generating function, the rank-crank PDE, Pólya cycle index polynomials, and finally, divisibility properties of multinomials. In Section 3 and Section 4, we prove our theorems. The last two sections are devoted to examples and open questions.

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2. PRELIMINARIES

2.1. Modular forms and quasimodular forms. The Eisenstein series G_k are modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$ for $k \geq 4$ even. If $k = 2$, then we need to add a non-holomorphic part to make G_2 modular. To be more precise,

$$\widehat{G}_2(\tau) := G_2(\tau) + \frac{1}{8\pi v} \tag{2.1}$$

transforms like a modular form of weight 2. The holomorphic part G_2 can be recovered by

$$\lim_{\bar{\tau} \rightarrow -i\infty} \widehat{G}_2(\tau) = G_2(\tau).$$

In general $f : \mathbb{H} \rightarrow \mathbb{C}$ is an *almost holomorphic modular form of weight $k \in \mathbb{Z}$ and depth $s \in \mathbb{N}_0$* , if the following conditions hold:

(1) We have, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau).$$

(2) We have,⁷ for some holomorphic functions $f_j : \mathbb{H} \rightarrow \mathbb{C}$ with $f_s \neq 0$,

$$f(\tau) = \sum_{j=0}^s \frac{f_j(\tau)}{v^j}.$$

(3) The function f grows at most polynomially in $\frac{1}{v}$ as $v \rightarrow 0$.

⁷The f_j here should not be confused with the Eisenstein-type series f_j .

By convention, the function zero is an almost holomorphic modular form of depth $-\infty$.

The holomorphic part f_0 is called a *quasimodular form of weight k and depth s* . Note that

$$f_0(\tau) = \lim_{\bar{\tau} \rightarrow -i\infty} f(\tau).$$

More concretely, for a quasimodular form g of weight k and depth s there exist holomorphic functions $g_j : \mathbb{H} \rightarrow \mathbb{C}$ for $j \in \{0, \dots, s\}$ with $g_0 = g$ such that, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$(c\tau + d)^{-k} g\left(\frac{a\tau + b}{c\tau + d}\right) = \sum_{j=0}^s g_j(\tau) \left(\frac{c}{c\tau + d}\right)^j.$$

Similarly, we say an analytic function $g^*(\tau, \bar{\tau})$ *transforms like a quasimodular form* if there exist real-analytic functions $g_j^*(\tau, \bar{\tau})$ so that, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$(c\tau + d)^{-k} g^*\left(\frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d}\right) = \sum_{j=0}^s g_j^*(\tau, \bar{\tau}) \left(\frac{c}{c\tau + d}\right)^j. \quad (2.2)$$

The space of all quasimodular forms is a free algebra with generators G_2, G_4 , and G_6 . It follows from Ramanujan's differential equations, [21, equations (1), (2)]

$$D(G_2) = -2G_2^2 + \frac{5}{6}G_4, \quad D(G_4) = -8G_2G_4 + \frac{7}{10}G_6, \quad D(G_6) = -12G_2G_6 + \frac{400}{7}G_4^2, \quad (2.3)$$

that this algebra is closed under differentiation. We also require the *Serre derivative* (see, e.g., [24, p. 48])

$$\vartheta_k := D + 2kG_2 \quad (2.4)$$

which acts on modular forms of weight k , preserving the algebra of modular forms.

The *Dedekind eta function*

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n)$$

is a modular form of weight $\frac{1}{2}$. It is not hard to deduce the following lemma.

Lemma 2.1. *For any holomorphic function f on \mathbb{H} , we have*

$$\eta D\left(\frac{f}{\eta}\right) = G_2 f + D(f).$$

2.2. Crank moments. Set $\zeta := e^{2\pi iz}$. By (1.2) and [23, equation (7)], the crank generating function can be expressed in terms of the Eisenstein series. Recall that $G_k = 0$ if k is odd.

Lemma 2.2. *We have*

$$C(\zeta; q) = \frac{\sin(\pi z)}{\pi z(q)_\infty} \exp\left(2 \sum_{k \geq 2} G_k(\tau) \frac{(2\pi iz)^k}{k!}\right).$$

Taking $\tau \rightarrow i\infty$, we obtain the following lemma, which was also observed in [2, Lemma 3.1].

Lemma 2.3. *We have*

$$\frac{\zeta^{\frac{1}{2}}}{\zeta - 1} = \frac{1}{2\pi iz} \exp\left(-\sum_{k \geq 2} \frac{B_k}{k} \frac{(2\pi iz)^k}{k!}\right).$$

Recall that the Bernoulli numbers B_n are defined as the constant terms of the Bernoulli polynomials $B_n(x)$ of degree n , which satisfy

$$\sum_{n \geq 0} B_n(X) \frac{t^n}{n!} = \frac{te^{Xt}}{e^t - 1}, \quad (2.5)$$

$$B_n(X + Y) = \sum_{k=0}^n \binom{n}{k} B_{n-k}(X) Y^k, \quad (2.6)$$

$$B'_n(X) = nB_{n-1}(X). \quad (2.7)$$

2.3. Mock modularity of the rank generating function. The rank generating function can be written as a Lerch sum

$$R(\zeta; q) = \frac{1 - \zeta}{(q)_\infty} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1 - \zeta q^n}. \quad (2.8)$$

Note that this closely resembles the following representation of the crank generating function

$$C(\zeta; q) = \frac{1 - \zeta}{(q)_\infty} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{1 - \zeta q^n}.$$

In Subsection 4.1 of [6], the first author defined (using different notation)

$$R^\#(z; \tau) := \left(\frac{R(\zeta; q)}{\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}} q^{-\frac{1}{24}} + \frac{1}{2} q^{-\frac{1}{6}} \sum_{\pm} \pm \zeta^{\mp 1} S(3z \pm \tau; 3\tau) \right) e^{12\pi^2 G_2(\tau) z^2}, \quad (2.9)$$

where $(\tau = u + iv, z = x + iy, u, v, x, y \in \mathbb{R})$

$$S(z; \tau) := \sum_{n \in \mathbb{Z} + \frac{1}{2}} \left(\operatorname{sgn}(n) - E \left(\left(n + \frac{y}{v} \right) \sqrt{2v} \right) \right) (-1)^{n - \frac{1}{2}} q^{-\frac{n^2}{2}} e^{-2\pi i n z},$$

with $E(y) := 2 \int_0^y e^{-\pi t^2} dt$. She showed, building on work of Zwegers [26], the following transformation:

Lemma 2.4. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{SL}_2(\mathbb{Z})$, we have

$$\eta \left(\frac{a\tau + b}{c\tau + d} \right) R^\# \left(\frac{z}{c\tau + d}; \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d) \eta(\tau) R^\#(z; \tau).$$

2.4. The rank-crank PDE. The rank-crank PDE of Atkin and Garvan [3, Theorem 1.1] relates the rank and crank generating functions by a differential equation. To state it, we define the *heat operator*⁸ (of index $-\frac{3}{2}$) as

$$H := 6q \frac{\partial}{\partial q} + \left(\zeta \frac{\partial}{\partial \zeta} \right)^2.$$

This heat operator maps Jacobi forms of weight $\frac{1}{2}$ to weight $\frac{5}{2}$ and does not change the index. We state a modified version of the rank-crank PDE which is more convenient for us.

⁸We point out that we re-normalize the standard heat operator by multiplying by a factor of $\frac{1}{4\pi^2}$ (see [14, p. 33] for more details).

Lemma 2.5. *We have*

$$2 \left(\frac{\zeta^{\frac{1}{2}}(q)_\infty C(\zeta; q)}{1 - \zeta} \right)^3 = (H + 6G_2(\tau)) \left(\frac{\zeta^{\frac{1}{2}}(q)_\infty R(\zeta; q)}{1 - \zeta} \right).$$

Proof. In [7, Theorem 14.28], the rank-crank PDE of Atkin–Garvan was formulated as

$$2\eta^2(\tau) \left(\frac{\zeta^{\frac{1}{2}}q^{-\frac{1}{24}}C(\zeta; q)}{1 - \zeta} \right)^3 = H \left(\frac{\zeta^{\frac{1}{2}}q^{-\frac{1}{24}}R(\zeta; q)}{1 - \zeta} \right).$$

Multiplying both sides by $\eta(\tau) = q^{\frac{1}{24}}(q)_\infty$ yields

$$2 \left(\frac{\zeta^{\frac{1}{2}}(q)_\infty C(\zeta; q)}{1 - \zeta} \right)^3 = \eta(\tau)H \left(\frac{1}{\eta(\tau)} \frac{\zeta^{\frac{1}{2}}(q)_\infty R(\zeta; q)}{1 - \zeta} \right).$$

We obtain the lemma, using the definition of H and Lemma 2.1 with

$$f(z; \tau) = \frac{\zeta^{\frac{1}{2}}(q)_\infty R(\zeta; q)}{1 - \zeta}. \quad \square$$

We also require the following lemma, inspired by the observation in [9] that the non-holomorphic part of a certain non-holomorphic Jacobi form, closely related to $R^\#$, is annihilated by the heat operator.

Lemma 2.6. *We have*

$$H \left(R^\#(z; \tau) e^{-12\pi^2 G_2(\tau) z^2} \right) = -H \left(\frac{\zeta^{\frac{1}{2}}R(\zeta; q)q^{-\frac{1}{24}}}{1 - \zeta} \right).$$

Proof. Using (2.9), it is enough to show

$$H \left(q^{-\frac{1}{6}} \zeta^{-1} S(3z + \tau; 3\tau) \right) = 0.$$

From (4.1) of [6], we have

$$q^{-\frac{1}{6}} \zeta^{-1} S(3z + \tau; 3\tau) = \sum_{n \in \mathbb{Z} - \frac{1}{6}} \left(\operatorname{sgn} \left(n - \frac{1}{3} \right) - E \left(\left(n + \frac{y}{v} \right) \sqrt{6v} \right) \right) (-1)^{n - \frac{5}{6}} q^{-\frac{3n^2}{2}} \zeta^{-3n}. \quad (2.10)$$

Thus the claim follows once we show that

$$H \left(\left(\operatorname{sgn} \left(n - \frac{1}{3} \right) - E \left(\left(n + \frac{y}{v} \right) \sqrt{6v} \right) \right) q^{-\frac{3n^2}{2}} \zeta^{-3n} \right) = 0.$$

Noting that $H(q^{-\frac{3n^2}{2}} \zeta^{-3n}) = 0$ and

$$H(fg) = H(f)g + fH(g) + 2 \left(\zeta \frac{\partial}{\partial \zeta} f \right) \left(\zeta \frac{\partial}{\partial \zeta} g \right),$$

it is enough to prove that

$$0 = H \left(E \left(\left(n + \frac{y}{v} \right) \sqrt{6v} \right) \right) - \frac{3n}{\pi i} \frac{\partial}{\partial z} E \left(\left(n + \frac{y}{v} \right) \sqrt{6v} \right) = \frac{3}{8\pi^2 v} [2\pi w E'(w) + E''(w)]_{w = (n + \frac{y}{v}) \sqrt{6v}}.$$

Thus we want to show that

$$2\pi x E'(x) + E''(x) = 0,$$

which holds since $E'(x) = 2e^{-\pi x^2}$ and $E''(x) = -4\pi x e^{-\pi x^2}$. □

2.5. Pólya cycle index polynomials. We require a result about Pólya cycle index polynomials in the case of symmetric group S_n , the set of permutations of the symbols x_1, x_2, \dots, x_n , see [22] and [1, Lemma 2.1] for more details.

Lemma 2.7 (Example 5.2.10 of [22]). *We have*

$$\sum_{n \geq 0} \sum_{\lambda \vdash n} \prod_{k=1}^n \frac{x_k^{m_k}}{m_k!} w^n = \exp \left(\sum_{k \geq 1} x_k w^k \right)$$

as a formal power series in w , where $\lambda = (1^{m_1}, 2^{m_2}, \dots, n^{m_n}) \vdash n$.

2.6. Divisibility of multinomials. Below, we require the following lemma regarding the divisibility of multinomial coefficients, which follows directly from Bézout's Lemma.

Lemma 2.8. *For $a_1, \dots, a_\ell \in \mathbb{N}$ with $\sum_{j=1}^{\ell} a_j = n$, we have*

$$\frac{n}{\gcd(a_1, \dots, a_\ell)} \mid \binom{n}{a_1, a_2, \dots, a_\ell}.$$

3. EISENSTEIN-TYPE SERIES AND THE PROOF OF THEOREMS 1.2 AND 1.6

3.1. Modularity and completion. Motivated by Lemma 2.2, we define the Eisenstein-type series f_k in terms of the rank generating function as follows:

$$R(\zeta; q) =: \frac{\sin(\pi z)}{\pi z(q)_\infty} \exp \left(2 \sum_{k \geq 1} f_k(\tau) \frac{(2\pi i z)^k}{k!} \right). \quad (3.1)$$

Remark 3.1. *Since R is invariant under $\zeta \mapsto \zeta^{-1}$, i.e., $z \mapsto -z$, we have $f_k = 0$ if k is odd.*

In the remainder of this section, we show that the f_k satisfies all of the properties in Theorem 1.2. To define their quasi-completions, we let

$$\begin{aligned} R^\circ(z, \bar{z}; \tau, \bar{\tau}) &:= 2i \sin(\pi z) q^{\frac{1}{24}} R^\#(z; \tau) e^{-12\pi^2 G_2(\tau) z^2} \\ &= R(\zeta; q) + \frac{1}{2} q^{-\frac{1}{8}} \left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}} \right) \sum_{\pm} \pm \zeta^{\mp 1} S(3z \pm \tau; 3\tau). \end{aligned} \quad (3.2)$$

Lemma 3.2. *We have*

$$\lim_{\bar{\tau} \rightarrow -i\infty} R^\circ(z, \bar{z}; \tau, \bar{\tau}) = R(\zeta; q).$$

Proof. By (3.2), we have

$$\lim_{\bar{\tau} \rightarrow -i\infty} R^\circ(z, \bar{z}; \tau, \bar{\tau}) = R(\zeta; q) + \frac{1}{2} q^{-\frac{1}{8}} \left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}} \right) \sum_{\pm} \pm \zeta^{\mp 1} \lim_{\bar{\tau} \rightarrow -i\infty} S(3z \pm \tau; 3\tau).$$

From equation (4.1) of [6], we directly obtain

$$\begin{aligned} &\lim_{\bar{\tau} \rightarrow -i\infty} q^{-\frac{1}{6}} \zeta^{-1} S(3z + \tau; 3\tau) \\ &= \sum_{n \in \mathbb{Z} - \frac{1}{6}} \left(\operatorname{sgn} \left(n - \frac{1}{3} \right) - \lim_{\bar{\tau} \rightarrow -i\infty} E \left(\left(n + \frac{y}{v} \right) \sqrt{6v} \right) \right) (-1)^{n - \frac{5}{6}} q^{-\frac{3n^2}{2}} \zeta^{-3n}. \end{aligned}$$

Now

$$\lim_{\bar{\tau} \rightarrow -i\infty} E\left(\left(n + \frac{y}{v}\right)\sqrt{6v}\right) = \lim_{v \rightarrow \infty} E\left(n\sqrt{6v} + \frac{\sqrt{6y}}{\sqrt{v}}\right) = \operatorname{sgn}(n).$$

As $\operatorname{sgn}(n - \frac{1}{3}) = \operatorname{sgn}(n)$ for $n \in \mathbb{Z} - \frac{1}{6}$ and $S(3z - \tau; 3\tau) = S(-3z + \tau; 3\tau)$, the claimed statement follows. \square

We next introduce quasi-completions f_k^* of f_k . First, we define

$$\frac{(q)_\infty R^\circ(z, \bar{z}; \tau, \bar{\tau})}{2i \sin(\pi z)} =: \frac{1}{2\pi iz} \exp\left(2 \sum_{k, \ell \geq 0} f_{k, \ell}^*(\tau, \bar{\tau}) \frac{(2\pi iz)^k}{k!} \frac{(2\pi i \bar{z})^\ell}{\ell!}\right). \quad (3.3)$$

We let⁹ $R^*(z; \tau)$ be the constant term of $R^\circ(z, \bar{z}; \tau, \bar{\tau})$ in the Taylor expansion in \bar{z} and set

$$f_k^*(\tau) := f_{k, 0}^*(\tau, \bar{\tau}).$$

In terms of these, we define¹⁰

$$\mathbb{F}(z; \tau) := \frac{(q)_\infty R^*(z; \tau)}{2i \sin(\pi z)} = \frac{1}{2\pi iz} \exp\left(2 \sum_{k \geq 0} f_k^*(\tau) \frac{(2\pi iz)^k}{k!}\right). \quad (3.4)$$

Note that $f_k^* = 0$ for k odd. We are now ready to prove Theorem 1.2 (2).

Lemma 3.3. *The function f_k^* is a quasi-completion of f_k . In particular, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{SL}_2(\mathbb{Z})$,*

$$f_k^*\left(\frac{a\tau + b}{c\tau + d}\right) = \begin{cases} (c\tau + d)^k f_k^*(\tau) & \text{if } k \neq 2, \\ (c\tau + d)^2 f_2^*(\tau) + \frac{3ic}{4\pi}(c\tau + d) & \text{if } k = 2. \end{cases}$$

Proof. We start by showing the transformation law of f_k^* . Combining (3.3) and (3.2) with (1.4) and Lemma 2.4, we obtain

$$\begin{aligned} & \frac{c\tau + d}{2\pi iz} \exp\left(2 \sum_{k, \ell \geq 0} f_{k, \ell}^*\left(\frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d}\right) \frac{\left(\frac{2\pi iz}{c\tau + d}\right)^k}{k!} \frac{\left(\frac{2\pi i \bar{z}}{c\bar{\tau} + d}\right)^\ell}{\ell!}\right) \\ &= \eta\left(\frac{a\tau + b}{c\tau + d}\right) R^\# \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) e^{-12\pi^2 G_2\left(\frac{a\tau + b}{c\tau + d}\right) \left(\frac{z}{c\tau + d}\right)^2} \\ &= (c\tau + d)\eta(\tau) R^\#(z; \tau) e^{-12\pi^2 \left(G_2(\tau) + \frac{ic}{4\pi(c\tau + d)}\right) z^2} \\ &= \frac{c\tau + d}{2\pi iz} \exp\left(2 \left(\frac{3ic}{4\pi(c\tau + d)} \frac{(2\pi iz)^2}{2!} + \sum_{k, \ell \geq 0} f_{k, \ell}^*(\tau, \bar{\tau}) \frac{(2\pi iz)^k}{k!} \frac{(2\pi i \bar{z})^\ell}{\ell!}\right)\right). \end{aligned}$$

Hence, we get

$$f_{k, \ell}^*\left(\frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d}\right) = \begin{cases} (c\tau + d)^k (c\bar{\tau} + d)^\ell f_{k, \ell}^*(\tau, \bar{\tau}) & \text{if } (k, \ell) \neq (2, 0), \\ (c\tau + d)^2 f_{2, 0}^*(\tau, \bar{\tau}) + \frac{3ic}{4\pi}(c\tau + d) & \text{if } (k, \ell) = (2, 0). \end{cases}$$

So the transformation formula for f_k^* follows.

⁹Note that here we suppress the dependence on $\bar{\tau}$.

¹⁰Again $\mathbb{F}(z; \tau)$ depends on $\bar{\tau}$.

We next show that

$$\lim_{\bar{\tau} \rightarrow -i\infty} f_k^*(\tau) = f_k(\tau). \quad (3.5)$$

By (3.3), Lemma 3.2, and (3.1) we have

$$\begin{aligned} \frac{1}{2\pi iz} \lim_{\bar{\tau} \rightarrow -i\infty} \exp\left(2 \sum_{k, \ell \geq 0} f_{k, \ell}^*(\tau, \bar{\tau}) \frac{(2\pi iz)^k}{k!} \frac{(2\pi i\bar{z})^\ell}{\ell!}\right) &= \frac{(q)_\infty}{2i \sin(\pi z)} \lim_{\bar{\tau} \rightarrow -i\infty} R^\circ(z, \bar{z}; \tau, \bar{\tau}) \\ &= \frac{(q)_\infty}{2i \sin(\pi z)} R(\zeta; q) = \frac{1}{2\pi iz} \exp\left(2 \sum_{k \geq 0} f_k(\tau) \frac{(2\pi iz)^k}{k!}\right). \end{aligned}$$

Comparing coefficients gives (3.5). \square

We also define

$$\widehat{\mathbb{F}}(z; \tau) := \frac{(q)_\infty R^*(z; \tau) e^{-\frac{3\pi z^2}{2v}}}{2i \sin(\pi z)} =: \frac{1}{2\pi iz} \exp\left(2 \sum_{k \geq 1} \widehat{f}_k(\tau) \frac{(2\pi iz)^k}{k!}\right). \quad (3.6)$$

Then, by the same argument used to prove Lemma 3.3, we obtain the following result.

Lemma 3.4. *For $k \in \mathbb{N}$, we have*

$$\widehat{f}_k(\tau) = \begin{cases} f_k^*(\tau) & \text{if } k \neq 2, \\ f_2^*(\tau) + \frac{3}{8\pi v} & \text{if } k = 2. \end{cases}$$

In particular

$$\widehat{f}_k\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \widehat{f}_k(\tau).$$

3.2. Limiting behavior of f_k . Next, we determine the behavior of $f_k(\tau)$ as $\tau \rightarrow i\infty$.

Lemma 3.5. *For $k \geq 2$, we have*

$$\lim_{\tau \rightarrow i\infty} f_k(\tau) = \lim_{\tau \rightarrow i\infty} f_k^*(\tau) = \lim_{\tau \rightarrow i\infty} \widehat{f}_k(\tau) = -\frac{B_k}{2k}.$$

Proof. Using (2.8), we first compute

$$\frac{(q)_\infty R(\zeta; q)}{2i \sin(\pi z)} = -\frac{\zeta^{\frac{1}{2}}(q)_\infty}{1 - \zeta} R(\zeta; q) = -\zeta^{\frac{1}{2}} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1 - \zeta q^n} \rightarrow -\frac{\zeta^{\frac{1}{2}}}{1 - \zeta}$$

as $\tau \rightarrow i\infty$. Thus, by (3.1),

$$\frac{\zeta^{\frac{1}{2}}}{\zeta - 1} = \frac{1}{2\pi iz} \exp\left(2 \sum_{k \geq 1} \lim_{\tau \rightarrow i\infty} f_k(\tau) \frac{(2\pi iz)^k}{k!}\right).$$

By Lemma 2.3, we obtain the claim for f_k .

To prove the claim for f_k^* , by (3.1) and (3.4), we have to show that

$$\lim_{\tau \rightarrow i\infty} \frac{(q)_\infty (R^*(z; \tau) - R(\zeta; q))}{2i \sin(\pi z)} = 0. \quad (3.7)$$

By (3.2), we have

$$R^*(z; \tau) - R(\zeta; q) = \frac{1}{2} q^{-\frac{1}{8}} \left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}} \right) \sum_{\pm} \pm \zeta^{\mp 1} S(3z \pm \tau; 3\tau).$$

Next, note that

$$S(3z \pm \tau; 3\tau) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \left(\operatorname{sgn}(n) - E \left(\left(n \pm \frac{1}{3} + \frac{y}{v} \right) \sqrt{6v} \right) \right) (-1)^{n-\frac{1}{2}} q^{-\frac{3n^2}{2}} e^{-2\pi i n(3z \pm \tau)}.$$

Using Lemma 1.7 of [26], as $\tau \rightarrow i\infty$ (so $v \rightarrow \infty$),

$$\operatorname{sgn}(n) - E \left(\left(n \pm \frac{1}{3} + \frac{y}{v} \right) \sqrt{6v} \right) \ll e^{-\pi \left(n \pm \frac{1}{3} + \frac{y}{v} \right)^2 6v}.$$

Thus $q^{-\frac{1}{8}} S(3z \pm \tau; 3\tau) \rightarrow 0$ as $\tau \rightarrow i\infty$, from which we obtain (3.7). The claim for \widehat{f}_k follows from the claim for f_k^* and Lemma 3.4. \square

3.3. Differential equations. Next, we look at the action of D on the f_k as well as the action of the raising and the lowering operator on its completions.

Proof of Theorem 1.6. (1) It is well-known, and follows by (2.3), that $D(G_\ell) \in \mathcal{F}$. Thus it remains to show the formula for $D(f_k)$ for $k \in 2\mathbb{N}$ which directly implies that $D(f_k) \in \mathcal{F}$. Using Theorem 2.5, Lemma 2.2, and (3.1), we obtain

$$\begin{aligned} & \frac{2}{(2\pi iz)^3} \exp \left(6 \sum_{k \geq 1} G_k(\tau) \frac{(2\pi iz)^k}{k!} \right) \\ &= \left(6q \frac{\partial}{\partial q} + \left(\zeta \frac{\partial}{\partial \zeta} \right)^2 + 6G_2(\tau) \right) \frac{1}{2\pi iz} \exp \left(2 \sum_{k \geq 1} f_k(\tau) \frac{(2\pi iz)^k}{k!} \right) \\ &= \left(\left(12 \sum_{k \geq 1} D(f_k(\tau)) \frac{(2\pi iz)^k}{k!} + 6G_2(\tau) \right) \frac{1}{2\pi iz} + \frac{2}{(2\pi iz)^3} + 2 \sum_{k \geq 1} f_k(\tau) \frac{(k-2)(2\pi iz)^{k-3}}{(k-1)!} \right. \\ & \quad \left. + 2 \left(-\frac{1}{(2\pi iz)^2} + 2 \sum_{k \geq 1} f_k(\tau) \frac{(2\pi iz)^{k-2}}{(k-1)!} \right) \sum_{\ell \geq 1} f_\ell(\tau) \frac{(2\pi iz)^{\ell-1}}{(\ell-1)!} \right) \exp \left(2 \sum_{k \geq 1} f_k(\tau) \frac{(2\pi iz)^k}{k!} \right). \end{aligned} \quad (3.8)$$

Multiplying by $\frac{(2\pi iz)^3}{2} \exp(-2 \sum_{k \geq 1} f_k(\tau) \frac{(2\pi iz)^k}{k!})$ and collecting all terms with $D(f_k)$ on one side and the rest on the other, we have

$$\begin{aligned} 6 \sum_{k \geq 1} D(f_k(\tau)) \frac{(2\pi iz)^{k+2}}{k!} &= -3G_2(\tau)(2\pi iz)^2 + \exp \left(2 \sum_{k \geq 1} (3G_k(\tau) - f_k(\tau)) \frac{(2\pi iz)^k}{k!} \right) - 1 \\ & \quad - \sum_{k \geq 1} f_k(\tau)(k-3) \frac{(2\pi iz)^k}{(k-1)!} - 2 \left(\sum_{k \geq 1} f_k(\tau) \frac{(2\pi iz)^k}{(k-1)!} \right)^2. \end{aligned} \quad (3.9)$$

Using Lemma 2.7 with $w = 2\pi iz$ and $x_k = \frac{2}{k!}(3G_k - f_k)$, we obtain

$$\exp\left(2 \sum_{k \geq 1} (3G_k(\tau) - f_k(\tau)) \frac{(2\pi iz)^k}{k!}\right) = \sum_{k \geq 0} \text{Tr}_k(\phi, 3G - f; \tau) (2\pi iz)^k.$$

Plugging this in (3.9) and extracting the coefficient of $(2\pi iz)^{k+2}$ on both sides, we get, for $k \geq 2$,

$$\frac{6}{k!} D(f_k(\tau)) = \text{Tr}_{k+2}(\phi, 3G - f; \tau) - \frac{k-1}{(k+1)!} f_{k+2}(\tau) - 2 \sum_{\substack{a, b \geq 0 \\ a+b=k}} \frac{1}{a!b!} f_{a+1}(\tau) f_{b+1}(\tau).$$

Multiplying with $\frac{k!}{6}$, yields the claim.

(2) By (3.2) and Lemma 2.6, we have

$$H\left(\frac{1}{\eta(\tau)} \frac{\zeta^{\frac{1}{2}}(q)_\infty R^\circ(z, \bar{z}; \tau, \bar{\tau})}{1-\zeta}\right) = H\left(\frac{1}{\eta(\tau)} \frac{\zeta^{\frac{1}{2}}(q)_\infty R(\zeta; q)}{1-\zeta}\right).$$

Using Lemma 2.1 and then Theorem 2.5, we obtain

$$(H + 6G_2(\tau)) \left(\frac{\zeta^{\frac{1}{2}}(q)_\infty R^\circ(z, \bar{z}; \tau, \bar{\tau})}{1-\zeta}\right) = 2 \left(\frac{\zeta^{\frac{1}{2}}(q)_\infty C(\zeta; q)}{1-\zeta}\right)^3.$$

Employing (3.3) for the left-hand side, we obtain

$$2 \left(\frac{(q)_\infty C(\zeta; q)}{2i \sin(\pi z)}\right)^3 = (H + 6G_2(\tau)) \left(\frac{1}{2\pi iz} \exp\left(2 \sum_{k, \ell \geq 0} f_{k, \ell}^*(\tau, \bar{\tau}) \frac{(2\pi iz)^k}{k!} \frac{(2\pi i \bar{z})^\ell}{\ell!}\right)\right).$$

Hence, using Lemma 2.2 and the definition of H , we obtain

$$\begin{aligned} & \frac{2}{(2\pi iz)^3} \exp\left(6 \sum_{k \geq 1} G_k(\tau) \frac{(2\pi iz)^k}{k!}\right) \\ &= \left(6q \frac{\partial}{\partial q} + \left(\zeta \frac{\partial}{\partial \zeta}\right)^2 + 6G_2(\tau)\right) \left(\frac{1}{2\pi iz} \exp\left(2 \sum_{k, \ell \geq 0} f_{k, \ell}^*(\tau, \bar{\tau}) \frac{(2\pi iz)^k}{k!} \frac{(2\pi i \bar{z})^\ell}{\ell!}\right)\right). \end{aligned}$$

Taking the constant terms from both sides with respect to \bar{z} gives

$$\begin{aligned} & \frac{2}{(2\pi iz)^3} \exp\left(6 \sum_{k \geq 1} G_k(\tau) \frac{(2\pi iz)^k}{k!}\right) \\ &= \left(6q \frac{\partial}{\partial q} + \left(\zeta \frac{\partial}{\partial \zeta}\right)^2 + 6G_2(\tau)\right) \left(\frac{1}{2\pi iz} \exp\left(2 \sum_{k \geq 0} f_k^*(\tau) \frac{(2\pi iz)^k}{k!}\right)\right). \end{aligned}$$

Now the shape of the above equation is exactly the same as in (3.8) except that f_k is replaced by f_k^* on the right-hand side. Hence, the same calculation as in part (1) gives that

$$\frac{6}{k!} D(f_k^*(\tau)) = \text{Tr}_{k+2}(\phi, 3G - f^*; \tau) - \frac{k-1}{(k+1)!} f_{k+2}^*(\tau) - 2 \sum_{\substack{a, b \geq 0 \\ a+b=k}} \frac{1}{a!b!} f_{a+1}^*(\tau) f_{b+1}^*(\tau).$$

Noting that $3G - f^* = 3\widehat{G} - \widehat{f}$, by Lemma 3.4, we have

$$\frac{6}{k!}D(f_k^*(\tau)) = \text{Tr}_{k+2}\left(\phi, 3\widehat{G} - \widehat{f}; \tau\right) - \frac{k-1}{(k+1)!}f_{k+2}^*(\tau) - 2 \sum_{\substack{a,b \geq 0 \\ a+b=k}} \frac{1}{a!b!}f_{a+1}^*(\tau)f_{b+1}^*(\tau).$$

From this, it is not hard to conclude the claim.

(3) Since $\widehat{\mathcal{F}}$ is generated by \widehat{f}_k for $k \in \mathbb{N}$, \widehat{G}_2, G_4 , and G_6 , it is enough to show that $L(\widehat{f}_k), L(\widehat{G}_2), L(G_4), L(G_6) \in \widehat{\mathcal{F}} \oplus \sqrt{v}|\eta|^2\widehat{\mathcal{F}}$ and prove the formula for $L(\widehat{f}_k)$. First, we have $L(G_4) = L(G_6) = 0 \in \widehat{\mathcal{F}} \oplus \sqrt{v}|\eta|^2\widehat{\mathcal{F}}$ since G_4 and G_6 are holomorphic. Next, by (2.1), we have $L(\widehat{G}_2) = -\frac{1}{8\pi} \in \widehat{\mathcal{F}} \oplus \sqrt{v}|\eta|^2\widehat{\mathcal{F}}$. Finally, we look at $L(\widehat{f}_k)$. We compute, using (3.6),

$$L(\widehat{\mathbb{F}}(z; \tau)) = 2\widehat{\mathbb{F}}(z; \tau) \sum_{k \geq 0} L(\widehat{f}_k(\tau)) \frac{(2\pi iz)^k}{k!}. \quad (3.10)$$

Using (3.6) again, the left-hand side can be written as

$$L\left(\frac{(q)_\infty R^*(z; \tau) e^{-\frac{3\pi z^2}{2v}}}{2i \sin(\pi z)}\right) = \frac{(q)_\infty}{2i \sin(\pi z)} \left(L(R^*(z; \tau)) + R^*(z; \tau) \frac{3\pi z^2}{2} \right) e^{-\frac{3\pi z^2}{2v}}.$$

Using the above for the left-hand side and (3.6) for the right-hand side of (3.10) gives

$$L(R^*(z; \tau)) = R^*(z; \tau) \sum_{k \geq 0} \left(2L(\widehat{f}_k(\tau)) + \frac{3}{4\pi} \delta_{k=2} \right) \frac{(2\pi iz)^k}{k!}. \quad (3.11)$$

We next compute, using (3.2),

$$L(R^*(z; \tau)) = \frac{1}{2}q^{-\frac{1}{8}} \left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}} \right) \sum_{\pm} \pm \zeta^{\mp 1} L(S(3z \pm \tau; 3\tau)). \quad (3.12)$$

We first consider the plus sign. Applying the lowering operator to (2.10) and using that by [6, p. 11], we have

$$L\left(E\left(\left(n + \frac{y}{v}\right) \sqrt{6v}\right)\right) = \sqrt{6v}^{\frac{3}{2}} \left(n - \frac{y}{v}\right) e^{-6\pi v \left(n + \frac{y}{v}\right)^2}$$

yields

$$L(S(3z + \tau; 3\tau)) = -\sqrt{6v}^{\frac{3}{2}} q^{\frac{1}{6}} \zeta e^{-\frac{6\pi y^2}{v}} \sum_{n \in \mathbb{Z} - \frac{1}{6}} \left(n - \frac{y}{v}\right) (-1)^{n - \frac{5}{6}} q^{\frac{3n^2}{2}} \zeta^{-3n}.$$

Next, we turn to the minus sign. Using that S is an even function and changing $z \mapsto -z$ in the above, yields

$$L(S(3z - \tau; 3\tau)) = L(S(-3z + \tau; 3\tau)) = -\sqrt{6v}^{\frac{3}{2}} q^{\frac{1}{6}} \zeta^{-1} e^{-\frac{6\pi y^2}{v}} \sum_{n \in \mathbb{Z} - \frac{1}{6}} \left(n + \frac{y}{v}\right) (-1)^{n - \frac{5}{6}} q^{\frac{3n^2}{2}} \zeta^{-3n}.$$

Plugging the above two equations back into (3.12) gives

$$\begin{aligned} L(R^*(z; \tau)) &= -\sqrt{\frac{3}{2}} v^{\frac{3}{2}} e^{-\frac{6\pi y^2}{v}} q^{\frac{1}{24}} \left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}} \right) \sum_{\pm} \pm \sum_{n \in \mathbb{Z} - \frac{1}{6}} \left(n \mp \frac{y}{v}\right) (-1)^{n - \frac{5}{6}} q^{\frac{3n^2}{2}} \zeta^{\pm 3n} \\ &= -i\sqrt{6v}^{\frac{3}{2}} e^{\frac{3\pi(z-\bar{z})^2}{2v}} q^{\frac{1}{24}} \sin(\pi z) \sum_{\pm} \pm \sum_{n \in \mathbb{Z} - \frac{1}{6}} \left(n \mp \frac{z - \bar{z}}{2iv}\right) (-1)^{n - \frac{5}{6}} q^{\frac{3n^2}{2}} \zeta^{\pm 3n}, \end{aligned}$$

using that $y = \frac{z-\bar{z}}{2i}$. Using (3.11) and (3.6), we obtain

$$\begin{aligned} \sum_{k \geq 0} \left(2L(\widehat{f}_k(\tau)) + \frac{3}{4\pi} \delta_{k=2} \right) \frac{(2\pi iz)^k}{k!} &= \frac{L(R^*(z; \tau))}{R^*(z; \tau)} \\ &= -\frac{i\sqrt{6}v^{\frac{3}{2}} e^{\frac{3\pi(z-\bar{z})^2}{2v}} q^{\frac{1}{24}} \sin(\pi z)}{R^*(z; \tau)} \sum_{\pm} \pm \sum_{n \in \mathbb{Z} - \frac{1}{6}} \left(n \mp \frac{z-\bar{z}}{2iv} \right) (-1)^{n-\frac{5}{6}} q^{\frac{3n^2}{2}} \zeta^{\pm 3n}. \end{aligned}$$

Collecting the constant terms in the Taylor expansions with respect to \bar{z} on both sides yields

$$\begin{aligned} \sum_{k \geq 0} \left(2L(\widehat{f}_k(\tau)) + \frac{3}{4\pi} \delta_{k=2} \right) \frac{(2\pi iz)^k}{k!} &= -\frac{i\sqrt{6}v^{\frac{3}{2}} e^{\frac{3\pi z^2}{2v}} q^{\frac{1}{24}} \sin(\pi z)}{R^*(z; \tau)} \sum_{\pm} \pm \sum_{n \in \mathbb{Z} - \frac{1}{6}} \left(n \mp \frac{z}{2iv} \right) (-1)^{n-\frac{5}{6}} q^{\frac{3n^2}{2}} \\ &= \frac{\sqrt{6}vz e^{\frac{3\pi z^2}{2v}} q^{\frac{1}{24}} \sin(\pi z)}{R^*(z; \tau)} \sum_{n \in \mathbb{Z} - \frac{1}{6}} (-1)^{n-\frac{5}{6}} q^{\frac{3n^2}{2}} = -iz \sqrt{\frac{3v}{2}} \frac{\eta(\tau)}{\widehat{\mathbb{F}}(z; \tau)} \sum_{n \in \mathbb{Z} - \frac{1}{6}} (-1)^{n-\frac{5}{6}} q^{\frac{3n^2}{2}}, \end{aligned}$$

using (3.6). The sum on the right-hand side equals $-\eta(-\bar{\tau})$, so the above becomes

$$iz \sqrt{\frac{3v}{2}} \frac{|\eta(\tau)|^2}{\widehat{\mathbb{F}}(z; \tau)} = -\pi \sqrt{6v} |\eta(\tau)|^2 z^2 \exp \left(-2 \sum_{k \geq 1} \widehat{f}_k(\tau) \frac{(2\pi iz)^k}{k!} \right),$$

by (3.6) again. Using Lemma 2.7 with $w = 2\pi iz$ and $x_k = -\frac{2}{k!} \widehat{f}_k(\tau)$, we conclude

$$\sum_{k \geq 0} \left(2L(\widehat{f}_k(\tau)) + \frac{3}{4\pi} \delta_{k=2} \right) \frac{(2\pi iz)^k}{k!} = \frac{\sqrt{3v}}{2\sqrt{2\pi}} |\eta(\tau)|^2 (2\pi iz)^2 \sum_{n \geq 0} \text{Tr}_n(\psi, \widehat{f}; \tau) (2\pi iz)^n.$$

Comparing the coefficient of $(2\pi iz)^k$ gives the claim for $k \geq 2$.

(4) The Serre derivative ϑ_k defined in (2.4) increases the weight of a modular object by 2 (note that the proof in [24, p. 48] goes through if f is non-holomorphic). We have

$$D + \frac{2k}{3} f_2^* - \vartheta_k = 2k \left(\frac{1}{3} f_2^* - G_2 \right).$$

By Lemma 3.3 and (1.4) this is a non-holomorphic modular form of weight 2. Hence $D + \frac{2k}{3} f_2^*$ maps a modular object of weight k to a modular object of weight $k+2$. \square

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Using (3.1) and Lemma 2.7 with $w = 2\pi iz$ and $x_k = \frac{2}{k!} f_k$, we find

$$R(\zeta; q) = \frac{\sin(\pi z)}{\pi z(q)_\infty} \sum_{k \geq 0} \text{Tr}_k(\phi, f; \tau) (2\pi iz)^k. \quad (3.13)$$

From (1.1) and (1.5) we deduce that

$$R(\zeta; q) = \sum_{k \geq 0} R_k(q) \frac{(2\pi iz)^k}{k!}. \quad (3.14)$$

Substituting $z \mapsto \frac{z}{2\pi i}$ and applying (3.14) and $\sin(-\frac{iz}{2}) = -i \sinh(\frac{z}{2})$ in (3.13), we obtain

$$\sum_{k \geq 0} R_k(q) \frac{z^k}{k!} = \frac{2 \sinh(\frac{z}{2})}{z(q)_\infty} \sum_{k \geq 0} \text{Tr}_k(\phi, f; \tau) z^k.$$

By Lemma 3.5 the first property is satisfied, by Lemma 3.3 the second, and by Theorem 1.6 (1) the third. \square

4. PROOF OF THEOREMS 1.3 AND 1.5

The following lemma rewrites the rank moments R_k in terms of the g_ℓ , defined by (1.7).

Lemma 4.1. *For $k \geq 1$, we have*

$$R_k(q) = \frac{2^{2-k}}{(q)_\infty} \sum_{\substack{\ell=2 \\ \ell \equiv k \pmod{2}}}^k \binom{k}{\ell-1} \left(g_\ell(\tau) + (2^{\ell-1} - 1) \frac{B_\ell}{2\ell} \right).$$

Proof. Both sides of the lemma are zero for k odd. Namely, since $N(m, n) = N(-m, n)$, we have that R_k is zero for k odd, and for ℓ odd, we have

$$(2^{\ell-1} - 1) \frac{B_\ell}{2\ell} + g_\ell(\tau) = 0. \quad (4.1)$$

So, we may assume that k is even. Recall that by [5, equation (2.12)] we have, for $k \geq 2$ even,

$$R_k(q) = \frac{2}{(q)_\infty} \sum_{n \geq 1} (-1)^{n+1} q^{\frac{n(3n-1)}{2}} (1 - q^n) \sum_{m \geq 0} m^k q^{nm}. \quad (4.2)$$

Distinguishing between n even and n odd yields

$$\begin{aligned} 2^{k-1} (q)_\infty R_k(q) &= \sum_{n \geq 1, m \geq 0} (2m)^k \left(-q^{n(6n+2m-1)} + q^{(2n-1)(3n+m-2)} + q^{n(6n+2m+1)} - q^{(2n-1)(3n+m-1)} \right) \\ &= \sum_{m \geq 3n \geq 3} (2m - 6n)^k \left(-q^{n(2m-1)} + q^{(2n-1)(m-2)} + q^{n(2m+1)} - q^{(2n-1)(m-1)} \right), \end{aligned}$$

making the change of variables $m \mapsto m - 3n$. Interchanging the role of n and m in the second and fourth sum the above becomes

$$\begin{aligned} - \sum_{m \geq 3n \geq 3} (2m - 6n)^k q^{n(2m-1)} + \sum_{n \geq 3m \geq 3} (2n - 6m)^k q^{(n-2)(2m-1)} \\ + \sum_{m \geq 3n \geq 3} (2m - 6n)^k q^{n(2m+1)} - \sum_{n \geq 3m \geq 3} (2n - 6m)^k q^{(n-1)(2m-1)}. \end{aligned}$$

Making the change of variables $n \mapsto n+2$ in the second sum, the change of variables $m \mapsto m-1$ in the third sum, and the change of variables $n \mapsto n+1$ in the fourth sum, the above becomes

$$\begin{aligned} - \sum_{2m-1 \geq 6n-1 \geq 5} (2m-1-6n+1)^k q^{n(2m-1)} + \sum_{2n+1 \geq 3(2m-1) \geq 3} (2n+1-3(2m-1))^k q^{n(2m-1)} \\ + \sum_{2m-1 \geq 6n+1 \geq 7} (2m-1-6n-1)^k q^{n(2m-1)} - \sum_{2n-1 \geq 3(2m-1) \geq 3} (2n-1-3(2m-1))^k q^{n(2m-1)} \\ = \sum_{\substack{m \geq 6n+1 \geq 7 \\ m \text{ odd}}} \left((m-6n-1)^k - (m-6n+1)^k \right) q^{nm} \end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{2n-1 \geq 3m \geq 3 \\ m \text{ odd}}} \left((2n-3m+1)^k - (2n-3m-1)^k \right) q^{nm} \\
= & \sum_{m \geq 6n+1 \geq 7} \left((m-6n-1)^k - (m-6n+1)^k \right) q^{nm} \\
& + \sum_{2n-1 \geq 3m \geq 3} \left((2n-3m+1)^k - (2n-3m-1)^k \right) q^{nm}. \quad (4.3)
\end{aligned}$$

We require the identity

$$(x+1)^k - (x-1)^k = 2 \sum_{\substack{1 \leq \ell \leq k-1 \\ 2|\ell}} \binom{k}{\ell} x^\ell = 2 \sum_{\substack{2 \leq \ell \leq k \\ 2|\ell}} \binom{k}{\ell-1} x^{\ell-1}.$$

Using this and the fact that k is even, we obtain that (4.3) equals

$$2 \sum_{\substack{\ell=2 \\ 2|\ell}}^k \binom{k}{\ell-1} \left(- \sum_{m \geq 6n+1 \geq 7} (m-6n)^{\ell-1} q^{nm} + \sum_{2n-1 \geq 3m \geq 3} (2n-3m)^{\ell-1} q^{nm} \right).$$

Now the lemma follows from the definition of g_ℓ . \square

The generating function of the g_ℓ is closely related to the rank generating function.

Lemma 4.2. *We have*

$$\frac{\pi z(q)_\infty}{\sin(\pi z)} R(\zeta; q) = 1 + 4 \sum_{k \geq 1} g_k(\tau) \frac{(\pi i z)^k}{(k-1)!}.$$

Proof. By setting $X = \frac{1}{2}$ and $t = -2\pi i z$ in (2.5) we have

$$\frac{\pi z}{\sin(\pi z)} = \sum_{n \geq 0} B_n \left(\frac{1}{2} \right) \frac{(2\pi i z)^n}{n!}. \quad (4.4)$$

Note that, by (1.5), $R_0(q) = \sum_{n \geq 0} \sum_{m \in \mathbb{Z}} N(m, n) q^n = \frac{1}{(q)_\infty}$ is the generating function of partitions. Hence, using (3.14) and Lemma 4.1, we obtain

$$\begin{aligned}
& \frac{\pi z(q)_\infty}{\sin(\pi z)} \left(R(\zeta; q) - \frac{1}{(q)_\infty} \right) \\
& = \sum_{n \geq 0} B_n \left(\frac{1}{2} \right) \frac{(2\pi i z)^n}{n!} \sum_{k \geq 1} 2^{2-k} \sum_{\substack{\ell=1 \\ \ell \neq k \pmod{2}}}^k \binom{k}{\ell} \left(g_{\ell+1}(\tau) + \frac{(2^\ell - 1) B_{\ell+1}}{2(\ell+1)} \right) \frac{(2\pi i z)^k}{k!} \\
& = \sum_{\substack{n \geq 0 \\ \ell \geq 1 \\ \ell \neq k \pmod{2}}} \sum_{\substack{k \geq \ell \\ k \neq \ell}} \binom{n}{k} \binom{k}{\ell} B_{n-k} \left(\frac{1}{2} \right) 2^{2-k} \left(g_{\ell+1}(\tau) + \frac{(2^\ell - 1) B_{\ell+1}}{2(\ell+1)} \right) \frac{(2\pi i z)^n}{n!},
\end{aligned}$$

where we make the change of variables $n \mapsto n - k$. Using (2.6), (2.7), and the fact that $B_m(\frac{1}{2}) = 0$ and $B_m(0) = -\frac{\delta_{m=1}}{2}$ for m odd, for $\ell \in \mathbb{N}$ odd and $n \in \mathbb{N}$ we find

$$\sum_{\substack{k=\ell \\ k \neq \ell \pmod{2}}}^n \binom{n}{k} \binom{k}{\ell} B_{n-k}\left(\frac{1}{2}\right) 2^{\ell-k} = -\binom{n}{\ell} B_{n-\ell}(0) = \begin{cases} \frac{n}{2} & \text{if } n = \ell + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Using the above and (4.1), we find

$$\frac{\pi z(q)_\infty}{\sin(\pi z)} \left(R(\zeta; q) - \frac{1}{(q)_\infty} \right) = \sum_{n \geq 1} \frac{n}{2^{n-2}} \left(\frac{(2^{n-1} - 1) B_n}{2n} + g_n(\tau) \right) \frac{(2\pi i z)^n}{n!}.$$

The result follows by adding $\frac{\pi z}{\sin(\pi z)}$ on both sides and using (4.4) and the fact that $B_n(\frac{1}{2}) = -(1 - 2^{1-n})B_n$. \square

Now, we are ready to prove Theorems 1.3 and 1.5.

Proof of Theorem 1.3. (1) Using Lemma 4.2 and then taking the derivative of (3.1) with respect to z yields

$$4\pi i \sum_{k \geq 1} f_k(\tau) \frac{(2\pi i z)^{k-1}}{(k-1)!} \exp\left(2 \sum_{k \geq 1} f_k(\tau) \frac{(2\pi i z)^k}{k!}\right) = 2\pi i \sum_{k \geq 0} \frac{k g_k(\tau)}{2^{k-2}} \frac{(2\pi i z)^{k-1}}{(k-1)!}. \quad (4.5)$$

Again using Lemma 4.2 we obtain

$$2 \sum_{k \geq 1} f_k(\tau) \frac{(2\pi i z)^{k-1}}{(k-1)!} + 2 \sum_{n, \ell \geq 1} \binom{n + \ell - 1}{n - 1} f_n(\tau) \frac{\ell g_\ell(\tau)}{2^{\ell-2}} \frac{(2\pi i z)^{n+\ell-1}}{(n + \ell - 1)!} = \sum_{k \geq 0} \frac{k g_k(\tau)}{2^{k-2}} \frac{(2\pi i z)^{k-1}}{(k-1)!}.$$

By extracting the coefficients of $(2\pi i z)^{k-1}$, we obtain (1).

(2) We send the exponential in (4.5) to the other side and apply Lemma 2.7 with $w = 2\pi i z$ and $x_k = -\frac{2f_k}{k!}$ to obtain

$$2 \sum_{k \geq 1} f_k(\tau) \frac{(2\pi i z)^{k-1}}{(k-1)!} = \sum_{n \geq 0} \frac{n g_n(\tau)}{2^{n-2}} \frac{(2\pi i z)^{n-1}}{(n-1)!} \sum_{m \geq 0} \text{Tr}_m(\psi, f; \tau) (2\pi i z)^m.$$

Comparing the coefficients of $(2\pi i z)^{k-1}$ gives (2). \square

Proof of Theorem 1.5. By (3.1), (3.14), and Lemma 2.3, we find that

$$\exp\left(2 \sum_{k \geq 1} \left(f_k(\tau) + \frac{B_k}{2k}\right) \frac{z^k}{k!}\right) = 1 + (q)_\infty \sum_{k \geq 1} R_k(q) \frac{z^k}{k!}.$$

Taking the logarithm and expanding the right-hand side formally, we deduce that

$$2 \sum_{k \geq 1} \left(f_k(\tau) + \frac{B_k}{2k}\right) \frac{z^k}{k!} = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \sum_{k_1, \dots, k_n \geq 1} z^{k_1 + \dots + k_n} \prod_{j=1}^n (q)_\infty R_{k_j}(q) \frac{1}{k_j!}.$$

Let Ω be the linear map on $\mathbb{C}[[z]]$ given by $\Omega(z^k) := k! z^k$ for $k \geq 0$. Applying Ω , we obtain

$$2 \sum_{k \geq 1} \left(f_k(\tau) + \frac{B_k}{2k}\right) z^k = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \mathcal{R}_n(z; q), \quad (4.6)$$

where we set

$$\mathcal{R}_n(z; q) := \sum_{k_1, \dots, k_n \geq 1} \binom{k_1 + \dots + k_n}{k_1, \dots, k_n} z^{k_1 + \dots + k_n} \prod_{j=1}^n (q)_\infty R_{k_j}(q).$$

To show that the Fourier coefficients of $f_k(\tau) + \frac{B_k}{2k}$ are integral, it suffices to show that these on the right-hand side of (4.6) are even integers. By (4.2) the Fourier coefficients of $(q)_\infty R_k(q)$ are even. Hence, those of $\frac{1}{n} \mathcal{R}_n(z; q)$ are even. Let $k_1, \dots, k_n \in \mathbb{N}$. There exists a unique partition λ of length n associated to these k_j (obtained by ordering the k_j in non-increasing order). Conversely, for a partition λ of length n , there are $\binom{n}{r_1(\lambda), r_2(\lambda), \dots}$ many n -tuples (k_1, \dots, k_n) with associated partition λ , where $r_m(\lambda)$ denotes the number of parts of size m in λ . Let \mathcal{P} be the set of partitions and $|\lambda| = \sum_j \lambda_j$ the size of $\lambda \in \mathcal{P}$. We replace the sum over k_1, \dots, k_n by a sum over partitions. Then, we have $\binom{k_1 + \dots + k_n}{k_1, \dots, k_n} = \binom{|\lambda|}{\lambda_1, \dots, \lambda_n}$ and

$$\mathcal{R}_n(z; q) = \sum_{\substack{\lambda \in \mathcal{P} \\ \ell(\lambda) = n}} \binom{|\lambda|}{\lambda_1, \dots, \lambda_n} \binom{n}{r_1(\lambda), r_2(\lambda), \dots} z^{|\lambda|} \prod_{j=1}^n (q)_\infty R_{\lambda_j}(q).$$

The claim follows once we show that

$$n \mid \binom{|\lambda|}{\lambda_1, \dots, \lambda_n} \binom{n}{r_1(\lambda), r_2(\lambda), \dots}. \quad (4.7)$$

Let d be the greatest common divisor of $r_1(\lambda), r_2(\lambda), \dots$. Since $\sum_{m \geq 1} r_m(\lambda) = n$, we have that $d \mid n$ and we write $n = rd$. Moreover, note that one can write the partition $(\lambda_1, \dots, \lambda_n)$ as $(\ell_1, \dots, \ell_1, \ell_2, \dots, \ell_2, \dots, \ell_r, \dots, \ell_r)$, where each ℓ_j is repeated d times. First, we show that

$$d \mid \binom{|\lambda|}{\lambda_1, \dots, \lambda_n}. \quad (4.8)$$

We factorize the multinomial coefficient as

$$\binom{|\lambda|}{\lambda_1, \dots, \lambda_n} = \binom{|\lambda|}{\frac{|\lambda|}{d}, \dots, \frac{|\lambda|}{d}} \binom{\frac{|\lambda|}{d}}{\ell_1, \dots, \ell_r}, \quad (4.9)$$

where $\frac{|\lambda|}{d}$ occurs d times in the first multinomial coefficient on the right-hand side. All of the multinomial coefficients are integers and by Lemma 2.8 with $n = |\lambda|$ and $a_j = \frac{|\lambda|}{d}$ for $j \in \{1, \dots, d\}$, we find that d divides the first factor in (4.9) and hence, (4.8) holds.

Again using Lemma 2.8, in this case with $a_m = r_m(\lambda)$ and $d = \gcd(a_1, \dots, a_\ell)$, we have $r \mid \binom{n}{r_1(\lambda), r_2(\lambda), \dots}$. Combining this with (4.8) gives (4.7). \square

Finally, we refine Lemma 3.5 by computing the first Fourier coefficients of f_k . In particular, the f_k are naturally normalized in the sense that the second Fourier coefficient is one.

Proposition 4.3. *For $k \geq 2$ even we have*

$$f_k(\tau) = -\frac{B_k}{2k} + q^2 + (2^k - 1)q^3 + O(q^4).$$

Proof. Recall that $(q)_\infty R_k(q) = 0$ for k odd. By (4.2), we have for k even

$$(q)_\infty R_k(q) = 2q^2 + 2(2^k - 1)q^3 + O(q^4).$$

Hence, we have $\prod_{j=1}^n (q)_\infty R_{k_j}(q) = O(q^4)$ if $n \geq 2$ and $k_j \in \mathbb{N}$ even. Therefore, by (4.6),

$$2 \sum_{k \geq 1} \left(f_k(\tau) + \frac{B_k}{2k} \right) z^k = 2 \sum_{\substack{k \geq 2 \\ k \text{ even}}} \left(q^2 + (2^k - 1) q^3 \right) z^k + O(q^4). \quad \square$$

Remark. Using a similar argument as in the proof of Proposition 4.3, one can show that the n -th Fourier coefficient of f_k is a linear combination of the k -th powers of $\{1, \dots, n-1\}$.

5. EXAMPLES

Here, we write down the first Fourier coefficients of the Eisenstein-type series f_k . We have

$$\begin{aligned} f_2(\tau) &= -\frac{1}{24} + q^2 + 3q^3 + 5q^4 + 7q^5 + 9q^6 + 10q^7 + 13q^8 + O(q^9), \\ f_4(\tau) &= \frac{1}{240} + q^2 + 15q^3 + 59q^4 + 139q^5 + 255q^6 + 406q^7 + 595q^8 + O(q^9), \\ f_6(\tau) &= -\frac{1}{504} + q^2 + 63q^3 + 635q^4 + 2827q^5 + 8199q^6 + 18550q^7 + 36043q^8 + O(q^9), \\ f_8(\tau) &= \frac{1}{480} + q^2 + 255q^3 + 6179q^4 + 53179q^5 + 253815q^6 + 844966q^7 + 2234875q^8 + O(q^9). \end{aligned}$$

Some examples for Theorem 1.6 (1) are given by

$$\begin{aligned} D(f_2) &= -f_2 G_2 - \frac{f_2^2}{2} - \frac{f_4}{12} + \frac{3G_2^2}{2} + \frac{G_4}{12}, \\ D(f_4) &= 6f_2^2 G_2 - 18f_2 G_2^2 - f_2 G_4 - f_4 G_2 - \frac{2}{3} f_2^3 - \frac{7f_4 f_2}{3} - \frac{f_6}{9} + 18G_2^3 + 3G_2 G_4 + \frac{G_6}{30}, \\ D(f_6) &= -60f_2^3 G_2 + 270f_2^2 G_2^2 + 15f_2^2 G_4 - 540f_2 G_2^3 + 30f_4 f_2 G_2 - 90f_2 G_2 G_4 - f_2 G_6 \\ &\quad - 45f_4 G_2^2 - f_6 G_2 - \frac{5f_4 G_4}{2} + 5f_2^4 - 5f_4 f_2^2 - \frac{11f_6 f_2}{3} - \frac{25f_4^2}{4} - \frac{f_8}{8} + 405G_2^4 \\ &\quad + \frac{21855G_4^2}{3652} + 135G_2^2 G_4 + 3G_2 G_6 - \frac{39G_8}{51128}, \\ D(f_8) &= 840f_2^4 G_2 - 5040f_2^3 G_2^2 - 280f_2^3 G_4 + 15120f_2^2 G_2^3 - 840f_4 f_2^2 G_2 + 2520f_2^2 G_2 G_4 \\ &\quad + 28f_2^2 G_6 - 22680f_2 G_2^4 + 2520f_4 f_2 G_2^2 - \frac{305970}{913} f_2 G_4^2 + 56f_6 f_2 G_2 - 7560f_2 G_2^2 G_4 \\ &\quad + 140f_4 f_2 G_4 - 168f_2 G_2 G_6 + \frac{39f_2 G_8}{913} - 2520f_4 G_2^3 - 84f_6 G_2^2 + 70f_4^2 G_2 - f_8 G_2 \\ &\quad - 420f_4 G_2 G_4 - \frac{14f_6 G_4}{3} - \frac{14f_4 G_6}{3} - 56f_2^5 + \frac{280}{3} f_4 f_2^3 - \frac{28}{3} f_6 f_2^2 - \frac{70}{3} f_4^2 f_2 - 5f_8 f_2 \\ &\quad - \frac{322f_4 f_6}{9} - \frac{2f_{10}}{15} + 13608G_2^5 + \frac{917910}{913} G_2 G_4^2 + 7560G_2^3 G_4 + 252G_2^2 G_6 \\ &\quad + \frac{19352886G_4 G_6}{1983949} + \frac{36751G_{10}}{1803590} - \frac{117G_2 G_8}{913}. \end{aligned}$$

6. QUESTIONS FOR FUTURE RESEARCH

We end by raising some open questions.

- (1) The three properties of the f_k given in Theorem 1.2 do not determine them uniquely. Therefore, we provide two recursive definitions of f_k in Theorem 1.3. It would be interesting to find “nice” properties that define an Eisenstein–type series uniquely. For example, the fact that G_k is a normalized Hecke eigenform of weight k which does not vanish at $i\infty$ determines it uniquely. As the coefficient of q in the Fourier expansion of the f_k vanishes by Proposition 4.3, one deduces that f_k cannot be a Hecke eigenform with respect to the usual action of the Hecke operators on q -series.
- (2) The functions f_2, f_4, f_6, \dots together with G_2, G_4 , and G_6 , do not seem to satisfy any algebraic relations. This has been verified numerically up to weight 24 (and also up to mixed weight 12). Is it indeed the case that the algebra \mathcal{F} is free?
- (3) What variations of the g_ℓ , in particular in the range of summation, are also mock modular? More concretely, for which $a, b \in \mathbb{N}$ is the function

$$\sum_{an-1 \geq bm \geq b} (an - bm)^{\ell-1} q^{nm} - \sum_{n-1 \geq abm \geq ab} (n - abm)^{\ell-1} q^{nm}$$

an Eisenstein–type series? Note that $a = 2$ and $b = 3$ yields the function g_ℓ in this paper. Are other choices of a and b also of particular interest? Do these functions, together with those in [25, p.15] and [19, equation (1.4)], form the first examples of a theory of (higher level) Eisenstein–type series?

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